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Applications of an identity of Andrews

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Abstract. In this paper, we give a bilateral form of an identity of Andrews, which is a generalization of the ${}_1\psi_1$ summation formula of Ramanujan. Using Andrews' identity, we deduce some new identities involving mock theta functions of second order and finally, we deduce some q-gamma, q-beta and eta function identities.

Keywords: q-Series; Mock theta functions; q-Gamma; q-Beta; Eta-functions

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1. INTRODUCTION AND STATEMENT OF RESULTS

In 1981, Andrews [2] has established the following identity

$$a^{-1}\sum_{n=0}^{\infty}\frac{(-q/a, AB/ab)_n}{(-B/a, -A/a)_{n+1}}(-b)^n - b^{-1}\sum_{n=0}^{\infty}\frac{(A, -aq/B)_n}{(-a, -A/b)_{n+1}}(-B/b)^n$$
$$= (a^{-1} - b^{-1})\frac{(A, B, bq/a, aq/b, q, AB/ab)_\infty}{(-b, -a, -A/b, -A/a, -B/b, -B/a)_\infty}, \quad |b|, |B/b| < 1, \qquad (1.1)$$

where as usual

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$$(a)_\infty := (a;q)_\infty := \prod_{n=0}^{\infty}(1-aq^n),$$
$$(a)_n := (a;q)_n := \frac{(a)_\infty}{(aq^n)_\infty}, \quad n \text{ is an integer},$$
$$(a_1,a_2,a_3,\ldots,a_m)_n = (a_1)_n(a_2)_n(a_3)_n\cdots(a_m)_n,$$
$$(a_1,a_2,a_3,\ldots,a_n)_\infty = (a_1)_\infty(a_2)_\infty(a_3)_\infty\cdots(a_n)_\infty.$$

This identity was proved using several summation and transformation formulae involving basic hypergeometric series. Putting $A = 0$, $a = -q/a$, $B = b/a$ and $b = -z$ in (1.1), we obtain the well-known ${}_1\psi_1$ summation formula of Ramanujan [9].

$$\sum_{n=-\infty}^{\infty}\frac{(a)_n}{(b)_n}z^n = \frac{(az)_\infty(q)_\infty(q/az)_\infty(b/a)_\infty}{(z)_\infty(b)_\infty(b/az)_\infty(q/a)_\infty}, \tag{1.2}$$

As indicated by Andrews [2] in his paper, Agarwal [1] and Kang [6] have proved (1.1) using the three term transformation formula of ${}_3\phi_2$-series [3, Equation (III.33), p. 364]. Recently, Liu [7] obtained the following equivalent form of (1.1) using (1.2) along with Roger-Fine identity by employing q-exponential operators.

Theorem 1.3. *If* $|a|, |b| < 1$, *then*

$$a^{-1}\sum_{k=0}^{\infty}\frac{(-q/a,cd/ab)_k}{(-c/a,-d/a)_{k+1}}(-b)^k - b^{-1}\sum_{k=0}^{\infty}\frac{(-q/b,cd/ab)_k}{(-c/b,-d/b)_{k+1}}(-a)^k$$
$$= (a^{-1}-b^{-1})\frac{(q,aq/b,bq/a,c,d,cd/ab)_\infty}{(-a,-b,-c/a,-c/b,-d/a,-d/b)_\infty}. \tag{1.3}$$

One can recover (1.1) from (1.3) by using Sears transformation for ${}_3\phi_2$-series [3, Equation (III.9), p. 359].

The main objective of this paper is to give a bilateral form of (1.3). As applications of (1.3) we derive some new identities involving mock theta functions of second order and also some q-gamma, q-beta and eta-function identities.

The q-gamma function $\Gamma_q(x)$, was introduced by Thomae [11] and later by Jackson [5] as

$$\Gamma_q(x) = \frac{(q)_\infty}{(q^x)_\infty}(1-q)^{1-x}, \quad 0<q<1. \tag{1.4}$$

q-Beta function is defined by

$$B_q(x,y) = (1-q)\sum_{n=0}^{\infty}\frac{(q^{n+1})_\infty}{(q^{n+y})_\infty}q^{nx}.$$

A relation between q-Beta function and q-gamma function is given by

$$B_q(x,y) = \frac{\Gamma_q(x)\Gamma_q(y)}{\Gamma_q(x+y)}. \tag{1.5}$$

The Dedekind eta function is defined by

$$\begin{aligned}\eta(\tau) &:= e^{\pi i\tau/12}\prod_{n=1}^{\infty}(1-e^{2\pi in\tau}), \quad Im(\tau)>0 \\ &:= q^{1/24}(q;q)_{\infty}, \quad \text{where } e^{2\pi i\tau}=q.\end{aligned} \tag{1.6}$$

In his last letter to Hardy [10], Ramanujan defined 17 new functions and called them 'mock' theta functions. For the definitions, classification and survey of recent developments in the theory of mock theta functions, one may refer to the paper by Gordan and McIntosh [4].

McIntosh [8] has defined the following mock theta functions and called them second order

$$A(q)=\sum_{n\geqslant 0}\frac{q^{(n+1)^2}(-q;q^2)_n}{(q;q^2)_{n+1}^2}, \tag{1.7}$$

$$B(q)=\sum_{n\geqslant 0}\frac{q^{n(n+1)}(-q^2;q^2)_n}{(q;q^2)_{n+1}^2}, \tag{1.8}$$

and

$$\mu(q)=\sum_{n\geqslant 0}\frac{(-1)^n q^{n^2}(q;q^2)_n}{(-q^2;q^2)_n^2}. \tag{1.9}$$

Ramanujan defines the general theta function $f(a,b)$ as

$$f(a,b):=\sum_{n=-\infty}^{\infty}a^{n(n+1)/2}b^{n(n-1)/2}=(-a,-b,ab;ab)_{\infty}, \quad |ab|<1.$$

Its special cases are

$$\phi(q):=f(q,q)=1+2\sum_{n=1}^{\infty}q^{n^2}=\frac{(-q;-q)_{\infty}}{(q;-q)_{\infty}}, \tag{1.10}$$

$$f(-q):=f(-q,-q^2)=\sum_{n=-\infty}^{\infty}(-1)^n q^{n(3n-1)/2}=(q;q)_{\infty}, \tag{1.11}$$

and

$$\chi(q):=(-q;q^2)_{\infty}. \tag{1.12}$$

q-shifted factorial identity is

$$(a)_{-n}=\frac{1}{(aq^{-n})_n}=\frac{(-q/a)^n}{(q/a)_n}q^{\binom{1}{2}}.$$

The sum of a very-well-poised ${}_6\psi_6$ series [3, Equation (II.33), p. 357] is given by

$$\begin{aligned}&{}_6\psi_6\left[\begin{matrix}q\sqrt{A}, & -q\sqrt{A}, & B, & C, & D, & E \\ \sqrt{A}, & -\sqrt{A}, & Aq/B, & Aq/C, & Aq/D, & Aq/E\end{matrix};q;\frac{qA^2}{BCDE}\right] \\ &\quad=\frac{(q,Aq,q/A,Aq/BC,Aq/BD,Aq/BE,Aq/CD,Aq/CE,Aq/DE)_{\infty}}{(q/B,q/C,q/D,q/E,Aq/B,Aq/C,Aq/D,Aq/E,qA^2/BCDE)_{\infty}}.\end{aligned} \tag{1.13}$$

Watson's transformation for a non-terminating very-well-poised ${}_8\phi_7$ series [3, Equation (III.17), p. 360] is given by

$$\begin{aligned}&\sum_{n=0}^{\infty}\frac{(A,qA^{1/2},-qA^{1/2},B,C,D,E,F)_n}{(q,A^{1/2},-A^{1/2},Aq/B,Aq/C,Aq/D,Aq/E,Aq/F)_n}\left(\frac{A^2q^2}{BCDEF}\right)^n\\&=\frac{(Aq,Aq/DE,Aq/DF,Aq/EF)_\infty}{(Aq/D,Aq/E,Aq/F,Aq/DEF)_\infty}\sum_{n=0}^{\infty}\frac{(D,E,F,Aq/BC)_n}{(q,Aq/B,Aq/C,DEF/A)_n}q^n. \qquad (1.14)\end{aligned}$$

2. Bilateral form of Identity (1.3)

In this Section, we give the bilateral form of (1.3) using Watson's transformation formula (1.14) for well-poised ${}_8\phi_7$-series.

Setting $A = aq/b$, $B = -aq/c$, $C = -aq/d$, $D = -q/b$, $E = q$ and $F = q^{-n}$ in (1.14), letting $n \to \infty$ and then multiplying the resulting identity throughout by $[(1 + c/b)(1 + d/b)]^{-1}$, we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-q/b,cd/ab)_k}{(-c/b,-d/b)_{k+1}}(-a)^k\\&=\sum_{k=0}^{\infty}\frac{(-q/b,-aq/c,-aq/d)_k}{(-a,-c/b,-d/b)_{k+1}}\left(1-\frac{aq^{2k+1}}{b}\right)(-1)^k q^{\binom{k}{2}}(cd/b)^k. \qquad (2.1)\end{aligned}$$

Interchange a and b in (2.1) to obtain

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(-q/a,cd/ab)_k}{(-c/a,-d/a)_{k+1}}(-b)^k\\&=\sum_{k=0}^{\infty}\frac{(-q/a,-bq/c,-bq/d)_k}{(-b,-c/a,-d/a)_{k+1}}\left(1-\frac{bq^{2k+1}}{a}\right)(-1)^k q^{\binom{k}{2}}(cd/a)^k. \qquad (2.2)\end{aligned}$$

From (2.1), (3.2) and (1.3), we have

$$\begin{aligned}&a^{-1}\sum_{k=0}^{\infty}\frac{(-q/a,-bq/c,-bq/d)_k}{(-b,-c/a,-d/a)_{k+1}}\left(1-\frac{bq^{2k+1}}{a}\right)(-1)^k q^{\binom{k}{2}}(cd/a)^k\\&-b^{-1}\sum_{k=0}^{\infty}\frac{(-q/b,-aq/c,-aq/d)_k}{(-a,-c/b,-d/b)_{k+1}}\left(1-\frac{aq^{2k+1}}{b}\right)(-1)^k q^{\binom{k}{2}}(cd/b)^k\\&=(a^{-1}-b^{-1})\frac{(q,aq/b,bq/a,c,d,cd/ab)_\infty}{(-a,-b,-c/a,-c/b,-d/a,-d/b)_\infty}, \qquad (2.3)\end{aligned}$$

which on some manipulations can be written as

$$\sum_{k=-\infty}^{\infty} \frac{(-q/a,-bq/c,-bq/d)_k}{(-b,-c/a,-d/a)_{k+1}} \left(1-\frac{bq^{2k+1}}{a}\right)(-1)^k q^{\binom{k}{2}} (cd/a)^k$$
$$= \frac{(q,a/b,bq/a,c,d,cd/ab)_\infty}{(-a,-b,-c/a,-c/b,-d/a,-d/b)_\infty}, |cd| < |a| < 1 \text{ and } |cd| < |b| < 1, \tag{2.4}$$

which is the required bilateral form of (1.3).

Remark 1. Letting $E \to \infty$ in (1.13) and then replacing A by bq/a, B by $-q/a$, C by $-bq/c$ and D by $-bq/d$, we obtain (3.4).

3. Some identities involve mock theta functions of second order

In this Section, we obtain some new identities involving mock theta functions of second order.

Theorem 3.1. *If $A(q)$, $B(q)$ and $\mu(q)$ are mock theta functions of second order as defined in 1.7,1.8 and 1.9 and $\phi(q)$, $f(q)$ and $\chi(-q)$ are Ramanujan's theta functions as defined in 1.10,1.11 and 1.12, then*

$$4A(q) + q^{-1}\mu(-q) = \chi^5(q)f(-q^2), \tag{3.2}$$
$$B(q) + B(-q) = 2\frac{f^5(-q^4)}{f^4(-q^2)}, \tag{3.3}$$
$$\mu(q) - A(-q) = \frac{\varphi(-q)f(-q)}{4f^2(-q^2)}. \tag{3.4}$$

Proof. Setting $a = -q/A$, $b = -Z$, $c = B/A$ and $d = C/A$ in (1.3) and then replacing the upper case letters to lower case, we obtain □

$$\sum_{k=0}^{\infty} \frac{(a,bc/azq)_k}{(b/q,c/q)_{k+1}} z^k - (q/az)\sum_{k=0}^{\infty} \frac{(q/z,bc/azq)_k}{(b/az,c/az)_{k+1}} (q/a)^k$$
$$= \frac{(q,az,q/az,b/a,c/a,bc/azq)_\infty}{(z,b/q,c/q,b/az,c/az,q/a)_\infty}. \tag{3.5}$$

Change q to q^2 and then replace b by bq, c by cq and z by $-zq/a$ in (3.5) and then let $a \to \infty$ in the resulting identity to obtain

$$\sum_{k=0}^{\infty} \frac{q^{k^2}(-bc/zq;q^2)_k}{(b/q,c/q;q^2)_{k+1}} z^k + (q/z)\sum_{k=0}^{\infty} \frac{q^{k^2+2k}(-bc/zq;q^2)_k}{(b/z,c/z;q^2)_{k+1}} z^{-k}$$
$$= \frac{(q^2,-qz,-q/z,-bc/zq;q^2)_\infty}{(b/q,c/q,b/z,c/z;q^2)_\infty}. \tag{3.6}$$

Putting $b = c = z = q^2$ in (3.6), we obtain

$$4A(q) + q^{-1}\sum_{k\geqslant 0}\frac{q^{k^2}(-q;q^2)_k}{(-q^2;q^2)_{k+1}} = \frac{(-q;q^2)^3_\infty (q^2;q^2)_\infty}{(-q^2;q^2)^2_\infty (q;q^2)^2_\infty}. \tag{3.7}$$

Using Euler's identity

$$(-q;q^2)_\infty = \frac{1}{(q;q^2)_\infty(-q^2;q^2)_\infty},$$

(1.11) and (1.12) in (3.7), we obtain (3.2) after some simplifications. Similarly, setting $b = c = q^2$ and $z = q$ in (3.6) using (1.11), we obtain (3.3) after some simplifications. Finally, set $b = c = -q$ and $z = -1$ in (3.6) using (1.10) and (1.11), we obtain (3.4) after some simplifications.

4. Some q-gamma, q-beta and eta function identities

In this Section, we deduce some interesting q-gamma, q-beta and eta function identities using (3.4), which can be written after changing the signs of 'a' and 'b' as

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(q/a, bq/c, bq/d)_k}{(b, c/a, d/a)_{k+1}}\left(1-\frac{bq^{2k+1}}{a}\right)q^{\binom{k}{2}}(cd/a)^k\\ &\quad -\tfrac{a}{b}\sum_{k=0}^{\infty}\frac{(q/b, aq/c, aq/d)_k}{(a, c/b, d/b)_{k+1}}\left(1-\frac{aq^{2k+1}}{b}\right)q^{\binom{k}{2}}(cd/b)^k\\ &= \frac{(q, a/b, bq/a, c, d, cd/ab)_\infty}{(a, b, c/a, c/b, d/a, d/b)_\infty}.\end{aligned} \tag{4.1}$$

Setting $a = q^x$, $b = q^y$, $c = q^{2x}$ and $d = q^{2y}$ in (4.1), we obtain

$$\begin{aligned}&\sum_{k=0}^{\infty}\frac{(q^{1-x}, q^{1-y}, q^{1+y-2x})_k}{(q^x, q^y, q^{2y-x})_{k+1}}(1-q^{2k+1+y-x})q^{\binom{k}{2}}q^{k(x+2y)}\\ &\quad - q^{y-x}\sum_{k=0}^{\infty}\frac{(q^{1-x}, q^{1-y}, q^{1+x-2y})_k}{(q^x, q^y, q^{2x-y})_{k+1}}(1-q^{2k+1+x-y})q^{(k2)}q^{k(2x+y)}\\ &= \frac{(q^{2x}, q^{2y}, q^{x+y}, q^{x-y}, q^{1+y-x}, q)_\infty}{(q^x, q^{2x-y}, q^{2y-x}, q^y, q^y, q^x)_\infty}.\end{aligned}$$

Using (1.4) and (1.5), we obtain after some simplifications,

$$\begin{aligned}\frac{B_q^2(x,y)}{B_q(2x,2y)} &= \frac{(1-q)(q^{2x-y}, q^{2y-x}, q^{x+y})_\infty}{(q^{x-y}, q^{1+y-x}, q^{2x+2y})_\infty}\\ &\times\left[\sum_{k=0}^{\infty}\frac{(q^{1-x}, q^{1-y}, q^{1+y-2x})_k}{(q^x, q^y, q^{2y-x})_{k+1}}(1-q^{2k+1+y-x})q^{\binom{k}{2}}q^{k(x+2y)}\right.\\ &\left. -q^{x-y}\sum_{k=0}^{\infty}\frac{(q^{1-x}, q^{1-y}, q^{1+x-2y})_k}{(q^x, q^y, q^{2x-y})_{k+1}}(1-q^{2k+1+x-y})q^{\binom{k}{2}}q^{k(2x+y)}\right]\end{aligned}$$

$0 < x, y < 1$ and $1/2y < x < 2y$.

Similarly, setting $a = q^x$, $b = q^y$ and $c = d = q^{x+y}$ in (4.1), using (1.5) we obtain after some simplifications,

$$B_q^3(x,y) = \frac{(q)_\infty^2(1-q)^3}{(q^{x-y}, q^{1+y-x})_\infty}\left[\sum_{k=0}^{\infty}\frac{(q^{1-x})_k^3}{(q^y)_{k+1}^3}(1-q^{2k+1+y-x})q^{\binom{k}{2}}q^{k(x+2y)}\right.$$
$$\left. - q^{y-x}\sum_{k=0}^{\infty}\frac{(q^{1-y})_k^3}{(q^x)_{k+1}^3}(1-q^{2k+1+x-y})q^{\binom{k}{2}}q^{k(2x+y)}\right]$$
$$0 < x, y < 1 \text{ and } 0 < x - y < 1.$$

Setting, $a = q^{2x}$, $b = q^x$ and $c = d = q^{4x}$ in (4.1), using (1.4), we obtain after some simplifications

$$\frac{\Gamma_q^3(2x)\Gamma_q^2(3x)}{\Gamma_q^2(4x)\Gamma_q(5x)} = \frac{(q)_\infty}{(q^{1-x})_\infty(1-q)^{x-2}}\left[\sum_{k=0}^{\infty}\frac{(q^{1-2x})_k(q^{1-3x})_k^2}{(q^x)_{k+1}(q^{2x})_{k+1}^2}(1-q^{2k+1-x})q^{\binom{k}{2}}q^{6kx}\right.$$
$$\left. - q^x\sum_{k=0}^{\infty}\frac{(q^{1-x})_k(q^{1-2x})_k^2}{(q^{2x})_{k+1}(q^{3x})_{k+1}^2}(1-q^{2k+1+x})q^{\binom{k}{2}}q^{7kx}\right] 0 < x < 1/3.$$

Finally, setting $a = q^{2x}$, $b = q^x$ and $c = d = q^{3x}$ in (4.1), using (1.4), we obtain after some simplifications,

$$\frac{\Gamma_q^2(x)\Gamma_q^3(2x)}{\Gamma_q^3(3x)} = \frac{(q)_\infty}{(q^{1-x})_\infty(1-q)^{2-x}}\left[\sum_{k=0}^{\infty}\frac{(q^{1-2x})_k^3}{(q^x)_{k+1}^3}(1-q^{2k+1-x})q^{\binom{k}{2}}q^{4kx}\right.$$
$$\left. - q^x\sum_{k=0}^{\infty}\frac{(q^{1-x})_k^3}{(q^{2x})_{k+1}^3}(1-q^{2k+1+x})q^{\binom{k}{2}}q^{5kx}\right] 0 < x < 1/2.$$

Changing $q \to q^2$ and setting $a = q$, $b = -q$ and $c = d = q^2$ in (4.1), using (1.4), we obtain after some simplifications,

$$\frac{\eta^6(4\tau)}{\eta^3(2\tau)} = \frac{1}{2}\left[\sum_{k=0}^{\infty}\frac{(-q;q^2)_k(1+q^{4k+2})}{(q;q^2)_{k+1}(1-q^{4k+2})}q^{k(k+1)}q^{2k} + \sum_{k=0}^{\infty}\frac{(-1)^k(q;q^2)_k(1+q^{4k+2})}{(-q;q^2)_{k+1}(1-q^{4k+2})}q^{k(k+1)}q^{2k}\right].$$

Similarly, changing $q \to q^2$ and setting $a = q$, $b = -q$, $c = q^2$ and $d = q^3$ in (4.1), using (1.6), we obtain after some simplifications,

$$\frac{\eta^2(4\tau)}{\eta(2\tau)} = \sum_{k=0}^{\infty}\frac{(-q^2;q^2)_{k-1}(1+q^{4k+2})}{(q^2;q^2)_{k+1}(1-q^{4k+2})}q^{k(k+1)}q^{3k}.$$

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